

Choosing Products in Social Networks

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Abstract

We study the consequences of adopting products by agents who form a social network. To this end we use the threshold model introduced in [2], in which the nodes influenced by their neighbours can adopt one out of several alternatives, and associate with such each social network a strategic game between the agents. The possibility of not choosing any product results in two special types of (pure) Nash equilibria.

We show that such games may have no Nash equilibrium and that determining an existence of a Nash equilibrium, also of a special type, is NP-complete. The situation changes when the underlying graph of the social network is a DAG, a simple cycle, or has no source nodes. For these three classes we determine the complexity of an existence of (a special type of) Nash equilibria and clarify the status of the finite improvement property (FIP). We also introduce the property of the uniform FIP which is satisfied when the underlying graph is a simple cycle.

Finally, we explain how these results can be used to analyze consequences of the addition of new products to a social network. In particular we show that in some cases such an addition can permanently destroy market stability.

1 Introduction

1.1 Background

Social networks are a thriving interdisciplinary research area with links to sociology, economics, epidemiology, computer science, and mathematics. A flurry of numerous articles and recent books, see, e.g., [6], testifies to the relevance of this field. It deals with such diverse topics as epidemics, analysis of the connectivity, spread of certain patterns of social behaviour, effects of advertising, and emergence of ‘bubbles’ in financial markets.

One of the prevalent types of models of social networks are the *threshold models* introduced in [8] and [15]. In such a setup each node i has a threshold $\theta(i) \in (0, 1]$ and adopts a given in advance ‘item’ (which can be a disease, trend, or a specific product) when the total weight of incoming edges from the nodes

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that have already adopted this item exceeds the threshold. One of the most important issues studied in the threshold models has been that of the spread of an item, see, e.g., [14, 11, 5]. From now on we shall refer to an ‘item’ that is spread by a more specific name of a ‘product’.

In this context very few papers dealt with more than one product. One of them is [9] with its focus on the notions of compatibility and bilinguality that result when one adopts both available products at an extra cost. Another one is [3], where the authors investigate whether the algorithmic approach of [11] can be extended to the case of two products.

In [2] we introduced a new threshold model of a social network in which nodes (agents) influenced by their neighbours can adopt one out of *several* products. This model allowed us to study various aspects of the spread of a given product through a social network, in the presence of other products. We analysed from the complexity point of view the problems of determining whether adoption of a given product by the whole network is possible (respectively necessary), and when a unique outcome of the adoption process is guaranteed. We also clarified for social networks without unique outcomes the complexity of determining whether a given node has to adopt some (resp. a given) product in some (resp. all) final network(s), and the complexity of computing the minimum and the maximum possible spread of a given product.

1.2 Motivation

Our interest here is in understanding and predicting the behaviour of the consumers (agents) who form a social network and are confronted with several alternatives (products). To carry out such an analysis we use the above model of [2] and associate with each such social network a natural strategic game. In this game the strategies of an agent are products he can choose. Additionally a ‘null’ strategy is available that models the decision of not choosing any product. The idea is that after each agent chose a product, or decided not to choose any, the agents assess the optimality of their choices comparing them to the choices made by their neighbours. This leads to a natural study of (pure) Nash equilibria, in particular of those in which some, respectively all, constituent strategies are non-null.

Social network games are related to graphical games of [10], in which the payoff function of each player depends only on a (usually small) number of other players. In this work the focus was mainly on finding mixed (approximate) Nash equilibria. However, in graphical games the underlying structures are undirected graphs. Also, social network games exhibit the following *join the crowd* property:

the payoff of each player weakly increases when more players choose his strategy.

Since these games are related to social networks, two natural special cases are of interest: when the underlying graph is a DAG or has no source nodes, with

the special case of a simple cycle. Such social networks correspond respectively to a hierarchical organization or to a ‘circle of friends’, in which everybody has a friend (a neighbour).

As noticed in a number of empirical studies, an abundance of choices may sometimes lead to wrong decisions. To quote from [7, page 38]:

The freedom-of-choice paradox. The more options one has, the more possibilities for experiencing conflict arise, and the more difficult it becomes to compare the options. There is a point where more options, products, and choices hurt both seller and consumer.

In our model we can analyse a similar phenomenon. Namely, we can explain how the availability of new products to the members of a social network can in some cases permanently destroy market stability.

1.3 Related work

There are a number of papers that focus on games associated with various forms of networks, see, e.g., an overview [16]. A more recent example is [1] that analyses a strategic game between players being firms who select nodes in an undirected graph in order to advertise competing products via ‘viral marketing’. However, in spite of the focus on similar questions concerning the existence and structure of Nash equilibria and on their reachability, from the technical point of view the games studied here seem to be unrelated to the games studied elsewhere.

Still, it is useful to mention the following phenomenon. When the underlying graph of a social network has no source nodes, the game always has a trivial Nash equilibrium in which no agent chooses a product. A similar phenomenon has been recently observed in [4] in the case of their network formation games, where such equilibria are called degenerate. Further, let us note that the ‘join the crowd’ property is exactly the opposite of the defining property of the congestion games with player-specific payoff functions introduced in [12]. In these game the payoff of each player weakly decreases when more players choose his strategy.

1.4 Plan of the paper

In the next section we recall the model of social networks introduced in [2] and define strategic games associated with these networks. Next, in Section 3 we show that in general Nash equilibria do not need to exist even if we limit ourselves to a special class of networks in which for each node all its neighbours have the same weight. We prove that determining an existence of a Nash equilibrium is NP-complete, also when we limit our attention to the two special types of equilibria.

Motivated by these results we consider in Section 4 three classes of social networks, the ones whose underlying graph is a DAG, a simple cycle, or, more generally, has no source nodes. For each class we determine the complexity of

the problem of existence of a (possibly special type of) Nash equilibrium. We also show that for these games the price of anarchy and of stability is unbounded.

Next, in Section 5 we study the finite improvement property (FIP) of [13] for these three classes of games. We show that each game associated with a social network whose underlying graph is a DAG has the FIP. When the underlying graph is a simple cycle only a weaker property holds. This property, that we call *the uniform FIP*, is of independent interest as it is stronger than that of being weakly acyclic (see [17] and [12]). We also show that when the underlying graph has no source nodes it can happen that no finite improvement path exists.

Finally, in Section 6 we summarize the obtained complexity results, explain how they can be used to study the consequences of introducing new products, and suggest some further research.

2 Preliminaries

2.1 Strategic games

Assume a set $\{1, \dots, n\}$ of players, where $n > 1$. A **strategic game** for n players, written as $(S_1, \dots, S_n, p_1, \dots, p_n)$, consists of a non-empty set S_i of **strategies** and a **payoff function** $p_i : S_1 \times \dots \times S_n \rightarrow \mathbb{R}$, for each player i .

Fix a strategic game $G := (S_1, \dots, S_n, p_1, \dots, p_n)$. We denote $S_1 \times \dots \times S_n$ by S , call each element $s \in S$ a **strategy profile**, denote the i th element of s by s_i , and abbreviate the sequence $(s_j)_{j \neq i}$ to s_{-i} . Occasionally we write (s_i, s_{-i}) instead of s .

We call a strategy s_i of player i a **best response** to a joint strategy s_{-i} of his opponents if $\forall s'_i \in S_i \ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i})$. Next, we call a strategy profile s a **Nash equilibrium** if each s_i is a best response to s_{-i} , that is, if

$$\forall i \in \{1, \dots, n\} \ \forall s'_i \in S_i \ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

Given a strategy profile s we call the sum $SW(s) = \sum_{j=1}^n p_j(s)$ the **social welfare** of s . When the social welfare of s is maximal we call s a **social optimum**. Recall that, given a finite game that has a Nash equilibrium, its **price of anarchy** (respectively, **price of stability**) is the ratio $\frac{SW(s)}{SW(s')}$ where s is a social optimum and s' is a Nash equilibrium with the lowest (respectively, highest) social welfare.

Following [13] a **path** in S is a sequence (s^1, s^2, \dots) of strategy profiles such that for every $k > 1$ there is a player i such that $s^k = (s'_i, s_{-i}^{k-1})$ for some $s'_i \neq s_i^{k-1}$. A path is called an **improvement path** if it is maximal and for all $k > 1$, $p_i(s^k) > p_i(s^{k-1})$, where i is the player who deviated from s^{k-1} . Finally, G has the **finite improvement property (FIP)** if every improvement path is finite. Obviously, if a game has the FIP, then it has a Nash equilibrium—it is the last element of each path.

2.2 Social networks

We are interested in specific strategic games defined over social networks. In what follows we focus on a model of the social networks recently introduced in [2].

Let $V = \{1, \dots, n\}$ be a finite set of **agents** and $G = (V, E)$ a weighted directed graph with $w_{ij} \in [0, 1]$ being the weight of the edge (i, j) . We often use the notation $i \rightarrow j$ to denote $(i, j) \in E$ and write $i \rightarrow^* j$ if there is a path from i to j in the graph G . Given a node i of G we denote by $\mathfrak{N}(i)$ the set of nodes from which there is an incoming edge to i . We call each $j \in \mathfrak{N}(i)$ a **neighbour** of i in G . We assume that for each node i such that $\mathfrak{N}(i) \neq \emptyset$, $\sum_{j \in \mathfrak{N}(i)} w_{ji} \leq 1$. An agent $i \in V$ is said to be a **source node** in G if $\mathfrak{N}(i) = \emptyset$.

Let \mathcal{P} be a finite set of alternatives or **products**. By a **social network** we mean a tuple $\mathcal{S} = (G, \mathcal{P}, P, \theta)$, where P assigns to each agent i a non-empty set of products $P(i)$ from which it can make a choice. For $i \in V$ and $t \in P(i)$ the **threshold function** θ yields a value $\theta(i, t) \in (0, 1]$. The threshold $\theta(i, t)$ should be viewed as agent i 's resistance level to adopt a product t .

Given a social network \mathcal{S} we denote by $source(\mathcal{S})$ the set of source nodes in the underlying graph G . One of the classes of social networks we shall study are the ones with $source(\mathcal{S}) = \emptyset$.

2.3 Social network games

Fix a social network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$. Each agent can adopt a product from his product set or choose not to adopt any product. We denote the latter choice by t_0 .

With each social network \mathcal{S} we associate a strategic game $\mathcal{G}(\mathcal{S})$ defined below. The idea is that the nodes simultaneously choose a product or abstain from choosing any. Subsequently each node assesses his choice by comparing it with the choices made by his neighbours. Formally, the game is defined as follows:

- the players are the agents,
- the set of strategies for player i is $S_i := P(i) \cup \{t_0\}$,
- For $i \in V$, $t \in P(i)$ and a strategy profile s , let $\mathcal{N}_i^t(s) := \{j \in \mathfrak{N}(i) \mid s_j = t\}$, i.e., $\mathcal{N}_i^t(s)$ is the set of neighbours of i who adopted in s the product t . The payoff function is defined as follows, where c is some given in advance positive constant:

$$\begin{aligned}
 & - \text{ for } i \in source(\mathcal{S}), \\
 & \quad p_i(s) = \begin{cases} 0 & \text{if } s_i = t_0 \\ c & \text{if } s_i \in P(i) \end{cases} \\
 & - \text{ for } i \notin source(\mathcal{S}), \\
 & \quad p_i(s) = \begin{cases} 0 & \text{if } s_i = t_0 \\ \sum_{j \in \mathcal{N}_i^t(s)} w_{ji} - \theta(i, t) & \text{if } s_i = t, \text{ for some } t \in P(i) \end{cases}
 \end{aligned}$$

The last entry in the payoff definition is motivated by the following considerations. In the case agent i is not a source node, his ‘satisfaction’ from a strategy profile depends positively from the accumulated weight (read: ‘influence’) of his neighbours who made the same choice as him, and negatively from his resistance level to adopt this product. So when his resistance level is high, it can happen that his payoff is negative. Of course, in such a situation not adopting any product is a better alternative.

By definition the payoff of each player depends only on the strategies chosen by his neighbours, so social network games are related to graphical games of [10]. However, the underlying dependence structure of a social network game is a directed graph and the presence of the special strategy t_0 available to each player makes these games more specific. Also, as already mentioned in Section 2, these games satisfy the ‘join the crowd’ property. Further, as we shall see in Section 4, in many social networks games Nash equilibria always exist.

In what follows for $t \in \mathcal{P} \cup \{t_0\}$ we use the notation \bar{t} to denote the strategy profile s where $s_j = t$ for all $j \in V$. This notation is legal only if for all agents i it holds that $t \in P(i)$.

The presence of the strategy t_0 motivates the introduction and study of special types of Nash equilibria in social network games we consider. We say that a Nash equilibrium s is

- **determined** if for all i , $s_i \neq t_0$,
- **non-trivial** if for some i , $s_i \neq t_0$
- **trivial** if for all i , $s_i = t_0$, i.e., $s = \bar{t_0}$.

3 Nash equilibria: general case

The first natural question that we address is that of the existence of Nash equilibria in the social network games. We establish the following result.

Theorem 1. *Deciding whether for a social network \mathcal{S} the game $\mathcal{G}(\mathcal{S})$ has a (respectively, non-trivial) Nash equilibrium is NP-complete.*

To prove it we first construct an example of a social network game with no Nash equilibrium and then use it to determine the complexity of the existence of Nash equilibria.

Example 2. Consider the social network given in Figure 1, where the product set of each agent is marked next to the node denoting the agent and the weights are labels on the edges. The source nodes are represented by the unique product in the product set.

We assume that the weights on the edges from the nodes marked by $\{t_1\}, \{t_2\}, \{t_3\}$ are w_1 , the weights on the edges forming the triangle are w_2 and that each threshold is a constant θ , where $\theta < w_1 < w_2$. So it is more profitable to a player residing on a triangle to adopt the product adopted by his neighbour

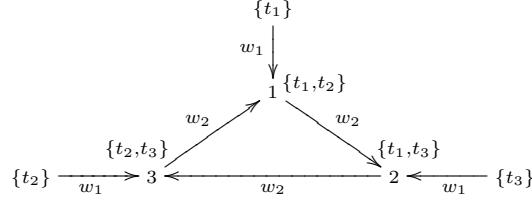


Figure 1: A social network with no Nash equilibrium

residing on a triangle, than by the other neighbour who is a source node. For convenience we represent each strategy profile as a tuple of products for players 1, 2 and 3.

It is easy to check that in the game associated with this social network no strategy profile is a Nash equilibrium. Indeed, each agent residing on the triangle can secure a payoff of at least $w_1 - \theta > 0$, so it suffices to analyze the strategy profiles in which t_0 is not used. There are in total eight such profiles. Here is their listing, where in each strategy profile we underline the strategy that is not a best response to the choice of other players: $(\underline{t_1}, t_1, t_2)$, $(t_1, \underline{t_1}, t_3)$, $(t_1, t_3, \underline{t_2})$, (t_1, t_3, t_3) , $(t_2, \underline{t_1}, t_2)$, $(t_2, \underline{t_1}, t_3)$, $(t_2, t_3, \underline{t_2})$, (t_2, t_3, t_3) . \square

Proof of Theorem 1.

As in [2] we use a reduction from the NP-complete PARTITION problem, which is: given n positive rational numbers (a_1, \dots, a_n) , is there a set S such that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i$? Consider an instance I of PARTITION. Without loss of generality, suppose we have normalized the numbers so that $\sum_{i=1}^n a_i = 1$. Then the problem instance sounds: does there is a set S such that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i = \frac{1}{2}$?

To construct the appropriate social network we employ the network given in Figure 1 and the social network given in Figure 2, where for each node $i \in \{1, \dots, n\}$ we set $w_{ia} = w_{ib} = a_i$, and assume that the threshold of the nodes a and b is constant and equal $\frac{1}{2}$.

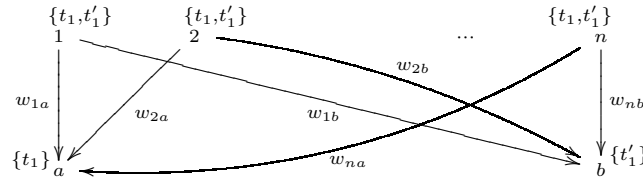


Figure 2: A social network related to the PARTITION problem

We use two copies of the social network given in Figure 1, one unchanged and the other in which the product t_1 is replaced by t'_1 and construct the desired social network \mathcal{S} by identifying the node a of the network from Figure 2 with the node marked by $\{t_1\}$ in the network in Figure 1, and the node b with the node marked by $\{t'_1\}$ in the modified version of the network from Figure 1.

Suppose now that a solution to the considered instance of the partition problem exists, that is for some set $S \in \{1, \dots, n\}$ we have $\sum_{i \in S} a_i = \sum_{i \notin S} a_i = \frac{1}{2}$. Consider the game $\mathcal{G}(S)$. Take the strategy profile formed by the following strategies:

- t_1 assigned to each node $i \in S$ in the network from Figure 2,
- t'_1 assigned to each node $i \in \{1, \dots, n\} \setminus S$ in the network from Figure 2,
- t_0 assigned to the nodes a, b and the nodes 1 in both versions of the network from Figure 1,
- t_3 assigned to the nodes 2, 3 in both versions of the networks from Figure 1 and the two nodes marked by $\{t_3\}$,
- t_2 assigned to the nodes marked by $\{t_2\}$.

We claim that this strategy profile is a non-trivial Nash equilibrium. Consider first the player (i.e, node) a . The accumulated weight of its neighbours who chose strategy t_1 is $\frac{1}{2}$, so its payoff after switching to the strategy t_1 is 0. Therefore t_0 is indeed a best response for player a . For the same reason t_0 is also a best response for player b . The analysis for the other nodes is straightforward and left to the reader.

Conversely, suppose that a strategy profile s is a Nash equilibrium in the game $\mathcal{G}(S)$. Then it is also a non-trivial Nash equilibrium. We claim that the strategy selected by the node a in s is t_0 . Otherwise, this strategy equals t_1 and the strategies selected by the nodes of the social network of Figure 1 form a Nash equilibrium in the game associated with this network. This yields a contradiction with our previous analysis of this social network.

So t_0 is a best response of the node a to the strategies of the other players chosen in s . This means that $\sum_{i \in \{1, \dots, n\} | s_i = t_1} w_{ia} \leq \frac{1}{2}$.

By the same reasoning t_0 is a best response of the node b to the strategies of the other players chosen in s . This means that $\sum_{i \in \{1, \dots, n\} | s_i = t'_1} w_{ib} \leq \frac{1}{2}$.

But $\sum_{i=1}^n a_i = 1$ and for $i \in \{1, \dots, n\}$, $w_{ia} = w_{ib} = a_i$, and $s_i \in \{t_1, t'_1\}$. So both above inequalities are in fact equalities. Consequently for $S := \{i \in \{1, \dots, n\} | s_i = t_1\}$ we have $\sum_{i \in S} a_i = \sum_{i \notin S} a_i$. In other words, there exists a solution to the considered instance of the partition problem.

Finally, given an instance of the PARTITION problem, the above social network \mathcal{S} can be constructed from it in a polynomial time. This proves the NP-hardness of the considered problem.

To prove that the problem lies in NP it suffices to notice that given a social network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ with n nodes checking whether a strategy profile is a non-trivial Nash equilibrium can be done by means of $n \cdot |\mathcal{P}|$ checks, so in polynomial time. \square

Focussing on determined Nash equilibria does not change the matters either.

Theorem 3. *Deciding whether for a social network \mathcal{S} the game $\mathcal{G}(\mathcal{S})$ has a determined Nash equilibrium is NP-complete.*

Proof. We use an instance of the PARTITION problem in the form of n positive rational numbers (a_1, \dots, a_n) , appropriately normalized, and the social network given in Figure 2.

Suppose now that a solution to the considered instance of the partition problem exists, that is for some set $S \in \{1, \dots, n\}$ we have $\sum_{i \in S} a_i = \sum_{i \notin S} a_i = \frac{1}{2}$. Take the strategy profile s formed by the following strategies:

- t_1 assigned to each node $i \in S$ and the node a ,
- t'_1 assigned to each node $i \in \{1, \dots, n\} \setminus S$ and the node b .

Then $p_a(t_1, s_{-i}) = \frac{1}{2} - \theta(a, t_1) = 0$, so $p_a(t_1, s_{-i}) \geq p_a(t_0, s_{-i})$. Analogously $p_b(t'_1, s_{-i}) \geq p_b(t_0, s_{-i})$. So s is a determined Nash equilibrium in the strategic game associated with the above social network.

Consider now a determined Nash equilibrium s in this game. Then $s_a = t_1$ and $p_a(t_1, s_{-i}) \geq p_a(t_0, s_{-i}) = 0$. So for $S := \{i \in \{1, \dots, n\} \mid s_i = t_1\}$ we have $\sum_{i \in S} a_i \geq \frac{1}{2}$. Analogously $\sum_{i \notin S} a_i \geq \frac{1}{2}$. So in both cases we have in fact equalities and hence there exists a solution to the considered instance of the partition problem. \square

4 Nash equilibria: special cases

In view of the fact that in general Nash equilibria may not exist we now consider social networks with special properties of the underlying directed graph. We focus on three natural classes.

4.1 Directed acyclic graphs

We consider here social networks whose underlying graph is a directed acyclic graph (DAG). Intuitively, such social networks correspond to hierarchical organizations. This restriction leads to a different outcome in the analysis of Nash equilibrium.

Given a DAG $G := (V, E)$, we use a fixed level by level enumeration $\text{rank}()$ of its nodes so that for all $i, j \in V$

$$\text{if } \text{rank}(i) < \text{rank}(j), \text{ then there is no path in } G \text{ from } j \text{ to } i. \quad (1)$$

Theorem 4. *Consider a social network \mathcal{S} whose underlying graph is a DAG.*

- (i) $\mathcal{G}(\mathcal{S})$ always has a non-trivial Nash equilibrium.
- (ii) Deciding whether $\mathcal{G}(\mathcal{S})$ has a determined Nash equilibrium is NP-complete.

Proof. (i) We assign to each node a strategy following the order determined by (1). Given a node we assign to it a best response to the sequence of strategies already assigned to all his neighbours. (By definition the strategies of other players have no influence on the choice of a best response). This result is also an immediate consequence of Theorem 13 which is proved in Section 5. (ii) follows from Theorem 3 since the social network from Figure 2 used in its proof is a DAG. \square

Note that when the underlying graph is a DAG all Nash equilibria are non-trivial. Further, in the procedure used above, in general more than one best response can exist. In that case multiple Nash equilibria exist.

Finally, we consider the price of anarchy and the price of stability for the considered class of games. The following simple result holds.

Theorem 5. *The price of anarchy and the price of stability for the games associated with the social networks whose underlying graph is a DAG is unbounded.*

Proof. Consider the social network depicted in Figure 3.

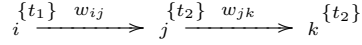


Figure 3: A social network with a high price of anarchy and stability

Choose an arbitrary value $r > 0$. Suppose first that the weights and thresholds exist such that $rc < 1$ and $w_{jk} - (\theta(j, t_2) + \theta(k, t_2)) > rc$. (Recall that the payoff of player i is c .)

The game associated with this network has a unique Nash equilibrium, namely the strategy profile (t_1, t_0, t_0) assigned to the sequence (i, j, k) of nodes. Its social welfare is c . In contrast, the social optimum is achieved by the profile (t_1, t_2, t_2) and equals

$$c + w_{jk} - (\theta(j, t_2) + \theta(k, t_2)) > rc.$$

So for every value $r > 0$ there is a social network whose game has price of anarchy and price of stability higher than r .

When such weights and thresholds do not exist, we modify the above social network as follows. First, we replace the node k by $\lceil rc + 1 \rceil$ nodes, all direct descendants of node j and each with the product set equal $\{t_2\}$. Then we choose the weights and the thresholds in such a way the sum of all these weights minus the sum of all the thresholds for the product t_2 exceeds rc . In the resulting game, by the same argument as above, both the price of anarchy and price of stability are higher than r . \square

4.2 Simple cycles

Next, we consider social networks whose underlying graph is a simple cycle. Clearly $\overline{t_0}$ is then a trivial Nash equilibrium. Checking whether there exists a non-trivial, respectively, determined, Nash equilibrium is easy.

To fix the notation suppose that the underlying graph is $1 \rightarrow 2 \rightarrow \dots \rightarrow n \rightarrow 1$. We assume that the counting is done in cyclic order within $\{1, \dots, n\}$ using the increment operation $i \oplus 1$ and the decrement operation $i \ominus 1$.

The payoff functions can then be rewritten as follows:

$$p_i(s) := \begin{cases} 0 & \text{if } s_i = t_0 \\ w_{i \ominus 1} - \theta(i, s_i) & \text{if } s_i = s_{i \ominus 1} \text{ and } s_i \in P(i) \\ -\theta(i, s_i) & \text{otherwise} \end{cases}$$

Theorem 6. *Consider a social network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ whose underlying graph is a simple cycle. There is a procedure that runs in time $\mathcal{O}(|\mathcal{P}| \cdot n)$, where n is the number of nodes in G , that decides whether $\mathcal{G}(\mathcal{S})$ has a non-trivial (respectively, determined) Nash equilibrium.*

Proof. Consider the following procedure

Procedure Nash(\mathcal{S})

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 $X := P(1);$ 
 $found := \mathbf{false};$ 
while  $X \neq \emptyset$  and not  $found$  do
  choose  $t \in X$ ;  $X := X \setminus \{t\}$ ;
   $i := 1$ ;
   $found := (w_{i \ominus 1} \geq \theta(i, t))$ ;
  while  $i \neq n$  and  $found$  do
     $i := i + 1$ ;
     $found := (t \in P(i)) \wedge (w_{i \ominus 1} \geq \theta(i, t))$ 
  od
od
return  $found$ 

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This procedure returns **true** if a non-trivial Nash equilibrium exists and **false** otherwise. Further, a Nash equilibrium is non-trivial iff it is determined, as in both cases the constituent strategies equal the same product. \square

Next, we consider the price of anarchy and the price of stability. We have the following counterpart of Theorem 5.

Theorem 7. *The price of anarchy and the price of stability for the games associated with the social networks whose underlying graph is a simple cycle is unbounded.*

Proof. Choose an arbitrary value $r > 0$ and let ϵ be such that $\epsilon < \frac{1}{2(r+1)}$. Consider the social network depicted in Figure 4.

We assume that

$$w_{12} - \theta(2, t_2) = 1 - \epsilon, \quad w_{21} - \theta(1, t_2) = -\epsilon, \quad w_{12} - \theta(2, t_1) = \epsilon, \quad w_{21} - \theta(1, t_1) = \epsilon.$$



Figure 4: A simple cycle with a high price of anarchy and stability

Then the social optimum is achieved in the profile (t_2, t_2) and equals $1 - 2\epsilon$. There are two Nash equilibria, (t_1, t_1) and the trivial one, with the respective social welfare 2ϵ and 0.

In the case of the price of anarchy we deal with the division by zero. We interpret the outcome as ∞ . The price of stability equals $\frac{1-2\epsilon}{2\epsilon}$, so is higher than r . \square

4.3 Graphs with no source nodes

Finally, we consider the case when the underlying graph $G = (V, E)$ of a social network \mathcal{S} has no source nodes, i.e., for all $i \in V$, $\mathfrak{N}(i) \neq \emptyset$. Intuitively, such a social network corresponds to a ‘circle of friends’: everybody has a friend (a neighbour). For such social networks we prove the following result.

Theorem 8. *Consider a social network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ whose underlying graph has no source nodes. There is a procedure that runs in time $\mathcal{O}(|\mathcal{P}| \cdot n^3)$, where n is the number of nodes in G , that decides whether $\mathcal{G}(\mathcal{S})$ has a non-trivial Nash equilibrium.*

The proof requires some characterization results that are of independent interest. The following concept plays a crucial role. Here and elsewhere we only consider subgraphs that are *induced* and identify each such subgraph with its set of nodes. (Recall that (V', E') is an induced subgraph of (V, E) if $V' \subseteq V$ and $E' = E \cap (V' \times V')$.)

We say that a (non-empty) strongly connected subgraph (in short, SCS) C_t of G is **self sustaining** for product t if for all $i \in C_t$,

- $t \in P(i)$,
- $\sum_{j \in \mathfrak{N}(i) \cap C_t} w_{ji} \geq \theta(i, t)$.

An easy observation is that if \mathcal{S} is a social network with no source nodes, then it always has a trivial Nash equilibrium, \bar{t}_0 . The following lemma states that for such networks every non-trivial Nash equilibrium satisfies a structural property which relates it to the set of self sustaining SCSs in the underlying graph. We use the following notation: for a profile s and product t , $\mathcal{A}_t(s) := \{i \in V \mid s_i = t\}$ and $P(s) := \{t \mid \exists i \in V \text{ with } s_i = t\}$.

Lemma 9. *Let $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ be a social network whose underlying graph has no source nodes. If $s \neq \bar{t}_0$ is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$ then for all products $t \in P(s) \setminus \{t_0\}$ and $i \in \mathcal{A}_t(s)$ there exists a self sustaining SCS $C_t \subseteq \mathcal{A}_t(s)$ for t and $j \in C_t$ such that $j \rightarrow^* i$.*

Proof. Suppose $s \neq \bar{t}_0$ is a Nash equilibrium. Take any product $t \neq t_0$ and an agent i such that $s_i = t$ (by assumption, at least one such t and i exists). Consider the set of nodes $\text{Pred} := \bigcup_{m \in \mathbb{N}} \text{Pred}_m$, where

- $\text{Pred}_0 := \{i\}$,
- $\text{Pred}_{m+1} := \text{Pred}_m \cup \bigcup_{j \in \text{Pred}_m} \mathcal{N}_j^t(s)$.

By construction for all $j \in \text{Pred}$, $s_j = t$ and $\mathcal{N}_j^t(s) \subseteq \text{Pred}$. Moreover, since s is a Nash equilibrium, we also have $\sum_{k \in \mathcal{N}_j^t(s)} w_{kj} \geq \theta(j, t)$.

Consider the partial ordering $<$ between the strongly connected components of the graph induced by Pred defined by: $C < C'$ iff $j \rightarrow k$ for some $j \in C$ and $k \in C'$. Take now some SCS C_t induced by a strongly connected component that is minimal in the $<$ ordering. Then for all $k \in C_t$ we have $\mathcal{N}_k^t(s) \subseteq C_t$ and hence $\mathcal{N}_k^t(s) \subseteq \mathfrak{N}(i) \cap C_t$. This shows that C_t is self sustaining.

Moreover, by the construction of Pred for all $j \in \text{Pred}$, and a fortiori for all $j \in C_t$, we also have $j \rightarrow^* i$. Since the choice of t and i was arbitrary, the claim follows. \square

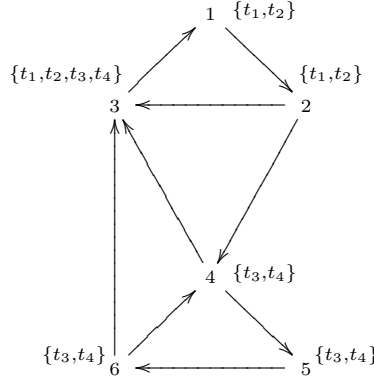


Figure 5: An equitable social network

The converse of Lemma 9 does not hold. Indeed, consider the equitable social network given in Figure 5. The product set of each agent is marked next to the node denoting the agent. Assume that the threshold of each node is a constant smaller than $\frac{1}{k}$, where k is the number of incoming edges. So each agent has a non-negative payoff when he adopts any product adopted by some of his neighbours. Consider the strategy profile s in which agents 1, 2 and 3 adopt product t_1 and agents 4, 5 and 6 adopt product t_3 , i.e., $s = (t_1, t_1, t_1, t_3, t_3, t_3)$. It follows from the definition that s satisfies the following condition of Lemma 9:

- for all products $t \in P(s) \setminus \{t_0\}$ and $k \in \mathcal{A}_t(s)$ there exists a self sustaining SCS $C_t \subseteq \mathcal{A}_t(s)$ for t and $j \in S_t$ such that $j \rightarrow^* k$.

However, s is not a Nash equilibrium since agent 3 has the incentive to deviate to product t_3 . Also, note that the profile $s' = (t_0, t_0, t_3, t_3, t_3, t_3)$ is a Nash equilibrium.

Using Lemma 9, we can now provide a necessary and sufficient condition for the existence of non-trivial Nash equilibria for the considered social networks.

Theorem 10. *Let $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ be a social network whose underlying graph has no source nodes. The strategy profile \bar{t}_0 is a unique Nash equilibrium in $\mathcal{G}(\mathcal{S})$ iff there does not exist a product t and a self sustaining SCS C_t for t in G .*

Proof. (\Leftarrow) Suppose there exists a strategy profile $s \neq \bar{t}_0$ such that s is a Nash equilibrium. Then by Lemma 9 there exists a self sustaining SCS C_t for every product $t \in P(s)$.

(\Rightarrow) Suppose there exists a self sustaining SCS C_t for a product t . Let R be the set of nodes reachable from C_t which eventually can adopt product t . Formally, $R := \bigcup_{m \in \mathbb{N}} R_m$ where

- $R_0 := C_t$,
- $R_{m+1} := R_m \cup \{j \mid t \in P(j) \text{ and } \sum_{k \in \mathfrak{N}(j) \cap R_m} w_{kj} \geq \theta(j, t)\}.$

Let s be the strategy profile such that for all $j \in R$, we have $s_j = t$ and for all $k \in V \setminus R$, we have $s_k = t_0$. It follows directly from the definition of R that s satisfies the following properties:

- (P1) for all $i \in V$, $s_i = t_0$ or $s_i = t$,
- (P2) for all $i \in V$, $s_i \neq t_0$ iff $i \in R$,
- (P3) for all $i \in V$, if $i \in R$ then $p_i(s) \geq 0$.

We show that s is a Nash equilibrium. Consider first any j such that $s_j = t$ (so $s_j \neq t_0$). By property (P2) $j \in R$ and by (P3) $p_j(s) \geq 0$. Since $p_j(s_{-j}, t_0) = 0 \leq p_j(s)$, player j does not gain by deviating to t_0 . Further, by (P1), for all $k \in \mathfrak{N}(j)$, $s_k = t$ or $s_k = t_0$ and therefore for all products $t' \neq t$ we have $p_j(s_{-j}, t') < 0 \leq p_j(s)$. Thus player j does not gain by deviating to any product $t' \neq t$ either.

Next, consider any j such that $s_j = t_0$. We have $p_j(s) = 0$ and from (P2) it follows that $j \notin R$. By the definition of R we have $\sum_{k \in \mathfrak{N}(j) \cap R} w_{kj} < \theta(j, t)$. Thus

$p_j(s_{-j}, t) < 0$. Moreover, for all products $t' \neq t$ we also have $p_j(s_{-j}, t') < 0$ for the same reason as above. So player j does not gain by a unilateral deviation. We conclude that s is a Nash equilibrium. \square

Next, for a product $t \in \mathcal{P}$, we define the set $X_t := \bigcap_{m \in \mathbb{N}} X_t^m$, where

- $X_t^0 := \{i \in V \mid t \in P(i)\},$
- $X_t^{m+1} := \{i \in V \mid \sum_{j \in \mathfrak{N}(i) \cap X_t^m} w_{ji} \geq \theta(i, t)\}.$

We need the following characterization result.

Theorem 11. *Let \mathcal{S} be a social network whose underlying graph has no source nodes. There exists a non-trivial Nash equilibrium in $\mathcal{G}(\mathcal{S})$ iff there exists a product t such that $X_t \neq \emptyset$.*

Proof. Suppose $\mathcal{S} = (G, \mathcal{P}, P, \theta)$.

(\Rightarrow) It follows directly from the definitions that if there is a self sustaining SCS C_t for product t then $C_t \subseteq X_t$. Suppose now that for all t , $X_t = \emptyset$. Then for all t , there is no self sustaining SCS for t . So by Theorem 10, \bar{t}_0 is a unique Nash equilibrium.

(\Leftarrow) Suppose there exists t such that $X_t \neq \emptyset$. Let s be the profile defined as follows:

$$s_i := \begin{cases} t & \text{if } i \in X_t \\ t_0 & \text{if } i \notin X_t \end{cases}$$

By the definition of X_t , for all $i \in X_t$, $p_i(s) \geq 0$. So no player $i \in X_t$ gains by deviating to t_0 (as then his payoff would become 0) or to a product $t' \neq t$ (as then his payoff would become negative since no player adopted t'). Also, by the definition of X_t and of the profile s , for all $i \notin X_t$ and for all $t' \in P(i)$, $p_i(t', s_{-i}) < 0$. Therefore, no player $i \notin X_t$ gains by deviating to a product t' either. It follows that s is a Nash equilibrium. \square

This theorem leads to a direct proof of the claimed result.

PROOF OF THEOREM 8. We use the following procedure for checking for the existence of a non-trivial Nash equilibrium.

Procedure Nash(\mathcal{S})

```

found := false;
while  $\mathcal{P} \neq \emptyset$  and  $\neg$ found do
  choose  $t \in \mathcal{P}$ ;
   $\mathcal{P} := \mathcal{P} - \{t\}$ ;
  compute  $X_t$ ;
  found := ( $X_t \neq \emptyset$ )
od
return found

```

On the account of Theorem 11 this procedure returns **true** if a non-trivial Nash equilibrium exists and **false** otherwise. To assess its complexity note that for a social network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ and a fixed product t , the set X_t can be constructed in time $\mathcal{O}(n^3)$, where n is the number of nodes in G . Indeed, each iteration of X_t^m requires at most $\mathcal{O}(n^2)$ comparisons and the fixed point is reached after at most n steps. In the worst case, we need to compute X_t for every $t \in \mathcal{P}$, so the procedure runs in time $\mathcal{O}(|\mathcal{P}| \cdot n^3)$. \square

The complexity changes in the case of determined Nash equilibria.

Theorem 12. *For a social network \mathcal{S} whose underlying graph has no source nodes, deciding whether $\mathcal{G}(\mathcal{S})$ has a determined Nash equilibrium is NP-complete.*

Proof. We modify the social network given in Figure 2 so that the graph has no source nodes. To this end we ‘twin’ each node $i \in \{1, \dots, n\}$ with a new node i' , also with the product set $\{t_1, t'_1\}$, by adding edges (i, i') and (i', i) . Additionally, we choose the weights $w_{ii'}$ and $w_{i'i}$ and the corresponding thresholds so that when i and i' adopt a common product, their payoff is positive.

For the so modified social network we can now repeat the proof of Theorem 3. \square

5 Finite improvement property

A natural question is whether the games for which we established the existence of a Nash equilibrium belong to some well-defined class of strategic games. In view of the examples of social networks given in Section 3, in general these games do not have the FIP. We now clarify whether the special classes of strategic games considered in Section 4 do have the FIP.

5.1 Directed acyclic graphs

In this section we analyze further the strategic games associated with the social networks whose underlying graph is a DAG.

Theorem 13. *For a social network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$, if G is a DAG then the game associated with this network has the FIP.*

Proof. We employ an enumeration $\text{rank}()$ of the nodes that satisfies property (1) from Subsection 4.1. Let $\Phi : S \rightarrow \mathbb{R}^n$ be a function which associates with every strategy profile an n -tuple of reals, defined as follows: $\Phi(s) := (p_{k(1)}(s), \dots, p_{k(n)}(s))$, where for all $i \in \{1, \dots, n\}$, $k(i) = \text{rank}^{-1}(i)$.

So for a profile s , the entries in $\Phi(s)$ are players’ payoffs for s , but rearranged according to the players’ rank. We use the notation $\Phi(s)[k]$ to denote the k -th entry in the tuple $\Phi(s)$. Finally, we use \preceq_L to denote the usual lexicographic ordering over n -tuples of reals and \prec_L to denote its strict order counterpart.

We argue that the value of the function Φ increases with each step of any improvement path. So suppose that strategy profiles s, s' form a step in an improvement path. By definition, for some player i and strategy s'_i , $p_i(s) < p_i(s'_i, s_{-i})$, i.e., $\Phi(s)[\text{rank}(i)] < \Phi(s'_i, s_{-i})[\text{rank}(i)]$. By (1) for all k such that $\text{rank}(k) < \text{rank}(i)$ the payoff of player k does not depend on player’s i strategy, and consequently $\Phi(s)[k] = \Phi(s'_i, s_i)[k]$. Therefore according to the lexicographic ordering we indeed have $\Phi(s) \prec_L \Phi(s'_i, s_{-i})$.

Therefore value of Φ strictly increases in the lexicographic ordering at each step of an improvement path. Since the domain of Φ is finite, it follows that the game has the FIP. \square

5.2 Simple cycles

The property that in the strategic game associated with a social network every improvement path is finite ceases to hold when the underlying graph has cycles.

Figure 6(a) gives an example. Take any threshold and weight functions which satisfy the condition that an agent gets positive payoff when he chooses the product picked by his unique predecessor in the graph. Figure 6(b) then shows an infinite improvement path. In each strategy profile, we underline the strategy that is not a best response to the choice of other players. Note that at each step of this improvement path a best response is used.

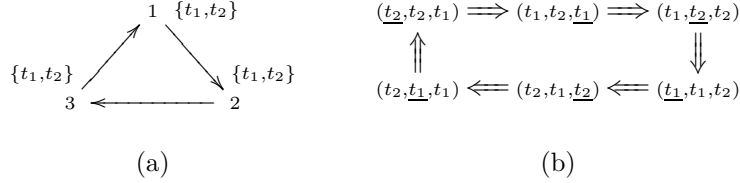


Figure 6: A social network with an infinite improvement path

This example can be readily generalized to an arbitrary simple cycle as soon as a strategy profile exists in which each player chose a product and in which at most one player plays a best response. Indeed, it suffices then to schedule the agents in the counterclockwise manner, with the player who played a best response scheduled last.

On the other hand, one can check that for any initial strategy profile there exists a finite improvement path. This is an instance of a more general result proved below.

By a **scheduler** we mean a function f that given a strategy profile s that is not a Nash equilibrium selects a player who does not play in s a best response. An improvement path $\rho = (s^1, s^2, \dots)$ **conforms** to a scheduler f if for all $k \geq 1$, $s^{k+1} = (s'_i, s^k_{-i})$, where $f(s^k) = i$. We say that a strategic game has the **uniform FIP** if there exists a scheduler f such that all improvement paths ρ which conform to f are finite. The property of having the uniform FIP is stronger than that of being **weakly acyclic** (see [17] and [12]) that only guarantees that for any strategy profile there exists a finite improvement path that starts at it.

Theorem 14. *Let \mathcal{S} be a social network such that the underlying graph is a simple cycle. Then the game $\mathcal{G}(\mathcal{S})$ has the uniform FIP.*

Proof. We use the scheduler f that chooses the smallest index i such that s_i is not a best response to s_{-i} . So this scheduler selects a player again if he did not switch to a best response. Therefore we can assume that each selected player immediately selects a best response.

Consider a strategy profile s taken from a ‘best response’ improvement path. Observe that for all k if $s_k \in P(k)$ and $p_k(s) \geq 0$ (so in particular if s_k is a best response to s_{-k}), then $s_k = s_{k \ominus 1}$. Consequently the following property holds for all $i > 1$:

$Z(i)$: if $f(s) = i$ and $s_{i-1} \in P(i-1)$ then for all $j \in \{n, 1, \dots, i-1\}$, $s_j = s_{i-1}$.

In words: if i is the first player who did not choose a best response and player's $i - 1$ strategy is a product, then this product is a strategy of every earlier player and of player n .

Along each 'best response' improvement path that conforms to f the value of $f(s)$ strictly increases until the path terminates or at certain stage $f(s) = n$. In the latter case if $s_{n-1} = t_0$, then the unique best response for player n is t_0 . Otherwise $s_{n-1} \in P(n - 1)$, so on the account of property $Z(n)$ all players' strategies equal the same product and the payoff of player n is negative (since $f(s) = n$). So the unique best response for player n is t_0 , as well.

This switch begins a new round with player 1 as the next scheduled player. Player 1 also switches to t_0 and from now on every consecutive player switches to t_0 , as well. The resulting path terminates once $n - 1$ player switches to t_0 . \square

Another scheduler f that we could use is the one that chooses the smallest index i such that $p_i(s) < 0$. Note that if $p_i(s) < 0$, then i does not play a best response in s . The argument is the same as for the above scheduler.

These two schedulers differ. Take for instance the social network given in Figure 6 and the initial strategy profile (t_2, t_2, t_1) . The first scheduler 'directs' the improvement path towards (t_1, t_1, t_1) , while the second one can direct it towards a trivial Nash equilibrium.

5.3 Graphs with no source nodes

Finally, we consider strategic games associated with arbitrary social networks whose underlying graph has no source nodes. Then it is possible that for some initial strategy profile *no* improvement path starting at it terminates, so in particular the game does not have the uniform FIP. To see this consider the social network given in Figure 7(a).

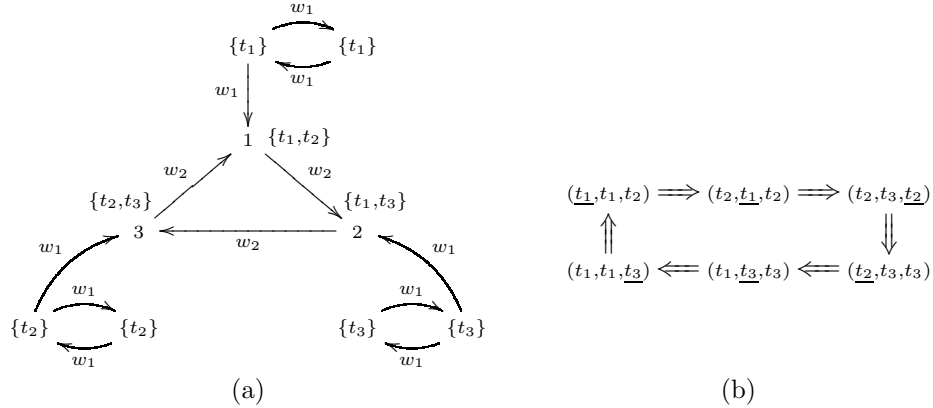


Figure 7: Another social network with an infinite improvement path

We assume that each threshold is a constant θ , where $\theta < w_1 < w_2$. Consider the strategy profile s , in which the nodes marked by $\{t_1\}$, $\{t_2\}$ and $\{t_3\}$ choose

the unique product in their product set and players 1, 2, and 3 choose t_1, t_1 and t_2 , respectively. For convenience, we denote s by (t_1, t_1, t_2) . This profile is not a Nash equilibrium since player 1 gains by deviating to t_2 . There is a unique improvement path starting at s and this path is infinite. This is shown in Figure 7(b). For each strategy profile in the figure, we underline the strategy that is not a best response.

6 Concluding remarks

6.1 Summary of the results

In this paper we studied the consequences of adopting products by agents who form a social network. To this end we analyzed a natural class of strategic games associated with the class of social networks introduced in [2]. We identified three natural types of (pure) Nash equilibria in these games: arbitrary, non-trivial, and determined, and focussed on games associated with four classes of social networks: arbitrary ones and those whose underlying graph is a DAG, or a simple cycle, or has no source nodes. The following table summarizes our complexity results, where we refer to the underlying graph with n nodes,

| Existence of a Nash equilibrium(NE) | | | | |
|-------------------------------------|-------------|---------------|---|---|
| NE | arbitrary | DAG | simple cycle | no source nodes |
| arbitrary | NP-complete | always exists | always exists | always exists |
| non-trivial | NP-complete | always exists | $\mathcal{O}(\mathcal{P} \cdot n)$ time | $\mathcal{O}(\mathcal{P} \cdot n^3)$ time |
| determined | NP-complete | NP-complete | $\mathcal{O}(\mathcal{P} \cdot n)$ time | NP-complete |

We also showed that the price of anarchy and the price of stability is unbounded, even if we limit ourselves to the social networks whose underlying graph is a DAG, respectively has no source nodes.

Finally, we studied the finite improvement property (FIP) and introduced a new notion of the games with the uniform FIP. The following table summarizes our results.

| Existence of the FIP property | | |
|-------------------------------|--------------|-----------------|
| DAG | simple cycle | no source nodes |
| yes | uniform FIP | no |

6.2 Adding new products

The proposed set up can be used to study consequences of the addition of, possibly new, products to a social network. To see this fix a social network and consider the corresponding strategic game. Suppose that the choices of the agents form a Nash equilibrium. This can be viewed as a stable situation ('nobody has an incentive to deviate'). Now suppose that some additional products become available to some players. An example is when a new product is introduced on the market. This may lead to some changes in agents' choices, as the new product may now become more attractive to some of them. A natural

question is whether the introduction of these additional products leads only to a temporary ‘perturbation’ of the market. We can frame this question using the notion of the FIP.

In the case when the underlying graph is a DAG the answer is positive. Indeed, it suffices to invoke Theorem 13, as the addition of the products does not affect the graph structure.

However, when the underlying graph has no source nodes, the answer is negative. To see this consider the social network given in Figure 7(a) where the product set of player 1 is $\{t_1\}$ instead of $\{t_1, t_2\}$. Then the strategy profile $s = (t_1, t_1, t_2)$ is a Nash equilibrium. Suppose we add the new product t_2 to player 1’s product set. With the expanded product set, the profile s fails to remain a Nash equilibrium since player 1 has an incentive to deviate to t_2 . Thus, as illustrated in Figure 7(b), there is an infinite improvement path starting at s . Moreover, this path is unique.

We conclude that in the case of social networks corresponding to hierarchical organizations, addition of new products leads to only a temporary perturbation. In contrast, in the case of the social networks corresponding to a group of friends addition of a new product can permanently destroy market stability.

6.3 Final comments

Let us mention that the results of this paper can be slightly generalized by using a more general notion of a threshold that would also depend on the set of neighbours who adopted a given product. In this more general setup for $i \in V$, $t \in P(i)$ and $X \subseteq \mathfrak{N}(i)$, the **threshold function** θ yields a value $\theta(i, t, X) \in (0, 1]$.

For the results to continue to hold one needs to assume that the threshold function satisfies the following **monotonicity** condition: if $X_1 \subseteq X_2$ then $\theta(i, t, X_1) \geq \theta(i, t, X_2)$. Intuitively, agent’s i resistance to adopt a product decreases when the number of its neighbours who adopted it increases. We decided not to use this definition for the sake of readability.

This work can be pursued in a couple of natural directions. One is the study of social networks with other classes of underlying graphs. Another is an investigation of the complexity results for other classes of social networks, in particular for the equitable social networks, i.e., networks in which the weight functions are defined as $w_{ij} = \frac{1}{|\mathfrak{N}(i)|}$ for all $i, j \in V$. Yet another is the analysis of the complexity of determining whether the FIP property or the uniform FIP property holds. In particular, we would like to determine the complexity of determining whether an addition of a product can lead to a permanent ‘perturbation’ of the market in the above sense.

Finally, we plan to study slightly different games, in which the players are obliged to choose a product, so the games in which the strategy t_0 is absent. Such games naturally correspond to situations in which the agents always choose a product, for instance a subscription for their mobile telephone. These games substantially differ from the ones considered here. For example, Nash equilibrium does not need to exist when the underlying graph is a simple cycle.

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